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On evolutionary approach for determining defuzzyfication operator

Witold Kosiński[1,2] and Urszula Markowska-Kaczmar[3]

[1] Polish-Japanese Institute of Information Technology
Research Center
ul. Koszykowa 86, 02-008 Warsaw, Poland
wkos@pjwstk.edu.pl
[2] Kazimierz Wielki University of Bydgoszcz
Institute of Environmental Mechanics and Applied Computer Science
ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland
[3] Institute of Applied Informatics, Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
Urszula.Markowska-Kaczmar@pwr.wroc.pl

Abstract. Ordered fuzzy numbers that make possible to deal with fuzzy inputs quantitatively, exactly in the same way as with real numbers, are defined together with four algebraic operations. For defuzzyfication operators that play the main role when dealing with fuzzy controllers and fuzzy inference systems an approximation formula is given. In order to determine it when a training set is given a dedicated evolutionary algorithm is presented. The form of the genotype composed of three types of chromosomes is given together with the fitness function. Genetic operators are also proposed.

1 Introduction

Fuzzy numbers [4, 24, 28] are very special fuzzy sets defined on the universe of all real numbers $\mathbf{R}$. They are of great importance in fuzzy systems. In applications the triangular and the trapezoidal (or triangular and trapezoidal shaped) fuzzy numbers are usually used or the so called (L, R)–numbers with two shape functions L and R proposed by Dubois and Prade [6] in 1978 as a restricted class of membership functions.

As long as one works with fuzzy numbers that possess continuous membership functions the two procedures: the extension principle and the α-cut and interval arithmetic method give the same results (cf. [2]). However, approximations of fuzzy functions and operations are needed if one wants to follow the extension principle and stay within (L, R)–numbers. It leads to some drawbacks as well as to unexpected and uncontrollable results of repeatedly applied operations [26, 27].

In most cases one assumes that membership function of a fuzzy number A satisfies convexity assumptions. It was Nguyen [23] who introducing his convex fuzzy numbers required from all α-cuts to be convex subsets of $\mathbf{R}$; the same he required from the support of A. At this stage it seems necessary to recall the both notions: if μ_A is the membership function of A then the α-cut of A is a (classical) set $A[\alpha] = \{x \in \mathbf{R} : \mu_A(x) \geq \alpha\}$, for each $\alpha \in [0, 1]$, and the support of A is the (classical) set $\text{supp } A = \{x \in \mathbf{R} : \mu_A(x) > 0\}$. One additionally assumes [2, 3, 5, 8, 23, 26] that the convex fuzzy number A has its core, i.e. the (classical) set of those $x \in \mathbf{R}$ for which its membership function $\mu_A(x) = 1$, which is not empty and its support is bounded.

However, even under those restrictions results of multiply operations on the convex fuzzy numbers are leading to the large grow of the fuzziness, and depend on the order of operations since the distributive law, which involves the interaction of addition and multiplication, does hold there.

This as well as other drawbacks have forced us to think about some generalization. Our main observation made in [14] was: a kind of quasi-invertibility of membership functions is crucial and one has to define arithmetic operations on their inverse parts to be in agreement with operations on the crisp real numbers.

Following the series of papers [12–19] a generalization of the classical concept of fuzzy numbers is made to define ordered fuzzy numbers and their algebra.

The main aim of the present paper is to deal with defuzzyfication operators (functionals) on ordered fuzzy numbers in Section 3. In the Section 4 the general approximation formula for such operators is presented. The paper brings an evolutionary algorithm which makes possible its determination.

2 Ordered fuzzy numbers

If μ_A is a membership function of a convex fuzzy number A two functions a_1, a_2 on $[0,1]$ can be defined that give lower and upper bounds of each α-cut of the membership function μ_A of the number A

$$A[\alpha] := \{x \in \mathbf{R} : \mu_A(x) \geq \alpha\} = [a_1(\alpha), a_2(\alpha)], \tag{1}$$

where boundary points are given for each $\alpha \in [0,1]$ by

$$a_1(\alpha) = \mu_A|_{incr}^{-1}(\alpha) \text{ and } a_2(\alpha) = \mu_A|_{decr}^{-1}(\alpha)\ . \tag{2}$$

In (2) the symbol $\mu_A|_{incr}^{-1}$ denotes the inverse function of the increasing part of the membership function $\mu_A|_{incr}$, the other symbol refers to the decreasing part $\mu_A|_{decr}$ of μ. Then we can see that the membership function μ_A of A is completely defined by two functions $a_1 : [0,1] \to \mathbf{R}$ and $a_2 : [0,1] \to \mathbf{R}$. In terms of them all arithmetic operations on the set of convex fuzzy numbers can be defined.

However, when the classical denotation for independent and dependent variables of the membership functions, namely x and y is used, we put $y = \alpha$ and use x for the denotation of values of the functions a_1 and a_2.

In this representation new, so-called ordered fuzzy numbers are defined which can be identified with pairs of continuous functions of y of the interval $[0,1]$ (compare (2)) with values x in $\mathcal{R}$.

Definition 1. *By an ordered fuzzy number A we mean an ordered pair (f,g) of functions such that $f, g : [0,1] \to \boldsymbol{R}$ are continuous.*

Notice, however, that in our definition we do not require that two continuous functions f and g are inverse functions of some membership function. Moreover, in generale membership function corresponding to A may not exist.

We call the corresponding elements: f – the up-part and g – the down-part of the fuzzy number A. The continuity of both parts implies their images are bounded intervals, say UP and $DOWN$, respectively (Fig. 1a). We have used symbols to mark boundaries for $UP = [l_A, 1_A^-]$ and for $DOWN = [1_A^+, p_A]$.

In general, the functions f, g need not to be invertible as functions of y, only continuity is required. If we assume, however, that: 1) they are monotonous, i.e. f is increasing, and g is decreasing, and 2) $f \leq g$ (pointwise), we may define the membership function $\mu(x) = f^{-1}(x)$, if $x \in [f(0), f(1)] = [l_A, 1_A^-]$, and $\mu(x) = g^{-1}(x)$, if $x \in [g(1), g(0)] = [1_A^+, p_A]$ and $\mu(x) = 1$ when $x \in [1_A^-, 1_A^+]$.

Notice that in general $f(1)$ needs not be less than $g(1)$. In this way we can reach improper intervals, which have been already discussed in the framework of the extended interval arithmetic by Kaucher in [7].

It is worthwhile to point out at this place that a large class of ordered fuzzy numbers represents the whole class of convex fuzzy numbers (OFN's) with continuous membership functions.

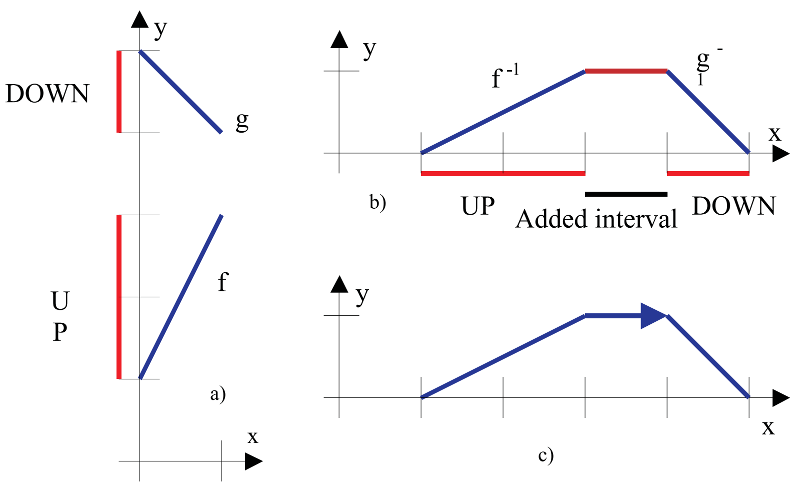

Fig. 1. a) Ordered fuzzy number, b) Ordered fuzzy number with membership function, c) Arrow denotes the order of inverted functions and the orientation

In Fig. 1 c) to the ordered pair of two continuous functions (here just two affine functions) f and g corresponds a membership function of a convex fuzzy number with an extra arrow which denotes the orientation of the closed curve formed below. This arrow shows that we are dealing with the ordered pair of functions and we can see that we have appointed to the convex fuzzy number in Fig. 1—and at the same time—to all ordered fuzzy numbers an extra feature, namely an orientation.

Notice that if some of the conditions 1) or 2) for f and g formulated above are not satisfied the construction of the classical membership function is not possible. However, in the previous paper [25] the '*corresponding*' membership function was defined. Here all algebraic operations are defined.

Definition 2. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers. The sum $C = A+B$, subtraction $C = A-B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula*

$$f_C(y) = f_A(y) \star f_B(y) \qquad \wedge \qquad g_C(y) = g_A(y) \star g_B(y) \tag{3}$$

where "$\star$" works for "$+$", "$-$", "$\cdot$", and "$\div$", respectively, and $A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero.

Notice that as long as we are adding ordered fuzzy numbers which possess their classical counterparts in the form of membership functions, and moreover, are of the same orientation, the results of addition is in agrement with the α-cut and interval arithmetic. However, this does not hold, in general, if the numbers have opposite orientations, for the result of addition may lead to improper intervals as far as some α-cuts are concerned. In this way we are close to the Kaucher arithmetic [7] with improper intervals, called by him directed intervals, i.e. such $[n, m]$ where n may be greater than m.

Thanks to this definition for any order fuzzy number A we will have $A - A = 0$, where 0 is the crisp zero.

Let $\mathcal{R}$ be a universe of all OFN's. Notice that this set is composed of all pairs of continuous functions defined on the closed interval $I = [0, 1]$ and can be identified with the linear space of real 2D-vector–valued functions defined on I with the norm of $\mathcal{R}$ as follows

$$||A|| = \max(\sup_{s \in I} |f_A(s)|, \sup_{s \in I} |g_A(s)|) \quad \text{if} \quad A = (f_A, g_A) \ .$$

Hence $\mathcal{R}$ is a Banach space. Neutral element of addition in $\mathcal{R}$ is a pair of constant function equal to crisp zero. It is also a Banach algebra with unity: the multiplication has a neutral element—the pair of two constant functions equal to one, i.e. the crisp one.

3 Defuzzyfication of ordered fuzzy numbers

Defuzzyfication is a main operation in fuzzy controllers and fuzzy inference systems [4, 20, 24] where fuzzy inference rules appear. If consequent parts of fuzzy rules are fuzzy, then a defuzzyfication process is needed, in the course of which to membership functions real numbers are attached. We know a number of defuzzyfication procedures for convex fuzzy numbers from the literature cf. [3, 24]. Then the problem arises – are the same defuzzyfication procedures applicable to ordered fuzzy numbers? The answer is partial positive: if the ordered fuzzy number is *proper* one, i.e. its membership relation is a function, then the same procedure can be applied. What to do, however, when the number is non-proper, i.e. the relation is by no means of functional type?

In the Banach space $\mathcal{R}$ equal to $C([0, 1]) \times C([0, 1])$ a general representation of linear and continuous functional on $\mathcal{R}$ can be obtained due to the Banach-Kakutami-Riesz representation theorem [1], which states that any linear and continuous functional $\bar{\phi}$ on a Banach space $C(S)$ of continuous functions defined on a compact topological space S is uniquely determined by a Radon (i.e. signed Borel) measure ν on S such that

$$\bar{\phi}(f) = \int_S f(s)\nu(ds) \quad \text{where } f \in C(S). \tag{4}$$

It is useful to remind that a Radon measure is a regular signed Borel measure (or differently: a difference of two positive Borel measures). A Boreal measure is a measure defined on σ-additive family of subsets of S which contains all open subsets.

In the case when the space S is an interval $[0,1]$ each Radon measure is represented by a Stieltjes integral $[1,21]$ with respect to a function of a bounded variation, i.e. for any continuous, linear functional $\bar{\phi}$ on $C([0,1])$ there is a function of bounded variation h such that

$$\bar{\phi}(f) = \int_0^1 f(s)dh(s) \quad \text{where } f \in C([0,1]). \tag{5}$$

Consequently, in the space of ordered fuzzy numbers $\mathcal{R}$ each bounded linear functional is given by a sum of two bounded, linear functionals defined on the space $C([0,1])$, i.e.

$$\phi(x_{up}, x_{down}) = \int_0^1 x_{up}(s)dh_1(s) + \int_0^1 x_{down}(s)dh_2(s) \tag{6}$$

where the pair of continuous functions $(x_{up}, x_{down}) \in \mathcal{R}$ represents an ordered fuzzy number and $h_1(s), h_2(s)$ are two functions of bounded variation defined on $[0,1]$.

Remark 1. Due to the general representation (6) and the functional representation (5) we can identified each linear and bounded functional on the space $\mathcal{R}$ with a pair (h_1, h_2) of functions of bounded variation.

From the above formula an infinite number of defuzzyfication procedures can be defined. The standard defuzzyfication procedure in terms of the area under membership relation can be defined; it is realized by a linear combinations of two Lebesgue measures of $[0,1]$. In the present case, however, the area is calculated in the y-variable, since the ordered fuzzy number is represented by a pair of continuous functions in y variable. Moreover to each point $s \in [0,1]$ a 'delta' (an atom) measure can be related, and such a measure represents a linear and bounded functional which realizes corresponding defuzzyfication procedure. Discussion of other linear functional as well as their non-linear generalization is done in another paper [10].

4 Approximation of defuzzyfication functional

In the recent paper [10] we have stated and proved the uniform approximation theorem concerning the defuzzyfication operators (functionals). Let us repeat its formulation. To this end let us make the following denotations.

Let $\mathcal{A} \subset \mathcal{R}$ be a compact subset[4] of the space of all ordered fuzzy numbers $\mathcal{R}$. By $\mathcal{G}$ we denote the set of all multivariant continuous functions defined on the appropriate cartesian product of the set of real numbers. In other words $F \in \mathcal{G}$ if there is a natural k such that $F : \mathbf{R}^k \to \mathbf{R}$ and is continuous in the natural norm of $\mathbf{R}^k$. By $\mathcal{D}$ we denote the set of all linear and continuous functionals defined on $\mathcal{A} \subset \mathcal{R}$. Here we could identify the set $\mathcal{D}$ with the adjoint space $\mathcal{R}^*$ since each continuous (bounded) and linear functional on the whole space $\mathcal{R}$ is also continuous, linear functional on each subspace, hence on the subset $\mathcal{A}$. Moreover, each continuous, linear, functional on a subspace $\mathcal{A} \subset \mathcal{R}$ can be extended to the whole space $\mathcal{R}$, thanks to the Hahn–Banach theorem [1].

If a function F of k–variables is from $\mathcal{G}$ and $\varphi_1, \varphi_2, \dots, \varphi_k \in \mathcal{D}$ then their superposition $F \circ (\varphi_1, \varphi_2, \dots, \varphi_k)$ is a function from $\mathcal{D}$ into $\mathbf{R}$, i.e. the functional

$$F \circ (\varphi_1, \varphi_2, \dots, \varphi_k) : \mathcal{D} \to \mathbf{R}, \text{ with } F \in \mathcal{G},\ \varphi_1, \varphi_2, \dots, \varphi_k \in \mathcal{D} \tag{7}$$

is a defuzzyfication operator, in general, nonlinear.

Theorem 1. *Let $\mathcal{A} \subset \mathcal{R}$ be a compact subset of the space of all ordered fuzzy numbers $\mathcal{R}$. Then the set $\mathcal{H}$ composed of all possible compositions (superpositions) of the type (7) where F is from $\mathcal{G}$ and $\varphi_1, \varphi_2, \dots, \varphi_k$ are from $\mathcal{D}$, with arbitrary k, is dense in the space of all continuous functionals from $\mathcal{R}$ into reals $\boldsymbol{R}$.*

Let us think how this theorem can help us in the following determination problem.

Problem A. Let to a given finite family of ordered fuzzy numbers, composed, say, of N numbers: $A_1, A_2, \dots, A_N$, values of a unknown continuous defuzzyfication operator be attached, say, $r_1, r_2, \dots, r_N$. The problem is to find the form of this operator.

In general the problem has no solution. However, we can look for its 'weak' solution in the approximate sense. In other words we can reformulate it having in mind Theorem 1.

[4] Notice that from the theorem of Ascoli-Arzelà [1] follows that a subset of $C([0,1])$ is compact if its elements are equi-continuous and equi-bounded.

Problem B. Let a finite set of training data be given in the form of N pairs: ordered fuzzy number and value (of action) of a defuzzyfication operator on it, i.e.

$$\mathsf{TRE} = \{(A_1, r_1), (A_2, r_2), \ldots, (A_N, r_N)\} \ . \tag{8}$$

For a given small ϵ find a continuous functional $\Psi : \mathcal{R} \to \mathbf{R}$ which approximates the values of the set TRE within the error smaller than ϵ. In other words, find Ψ defined on $\mathcal{R}$ such that

$$\max_{1 \le p \le N} |\Psi(A_p) - r_p| \le \epsilon \ , \ \text{where } (A_p, r_p) \in \mathsf{TRE} \ . \tag{9}$$

The second problem **B** may possess several solutions, however, one can look for one of them with the help of Theorem 1 and the classical result of the approximation theory known under the term of the Weierstrass theorem. This theorem states that each continuous function (of many variables) defined on a compact set can be approximated with a given accuracy by a polynomial (of many variables) of an appropriate order, i.e sufficiently high order.

In what follows we are assuming that the training set is not trivial, i.e. it possesses at least two different, in both positions, pairs. We are going to propose a method of finding a solution to Problem B in the form of a superposition of a polynomial (of many variables) with a number of linear functionals from $\mathcal{D}$. To this end the use of a specially dedicated evolutionary algorithm will be made.

The role of the polynomial is to approximate some nonlinear function F from the set $\mathcal{G}$. Unfortunately, Theorem 1 does not tell how many independent variables has the function F. On the other hand the cardinality of the family TRE gives the natural upper-bound on the number of independent variables of the unknown function F and consequently of the polynomial. Since the cardinality of the family is N the upper-bound on the order of the polynomial is $N-1$ if it were a polynomial of one variable, since N values $r_p, p = 1, 2, \ldots, N$ can be used to determine N coefficients $c_0, c_1, \ldots_{N-1}$ standing in the front of the corresponding powers of the variable. However, if it were polynomial of k variables then its order should be smaller, say $n < N$ since then the maximal number M of its coefficients is given by the combinatoric formula

$$M = \binom{n+k}{k} , \tag{10}$$

and it should not exceed N. Such a formula follows from the fact ([22], p.31), that the number of different solutions in the form of nonnegative integers of the inequalities

$$x_1 + x_2 + \cdots + x_k \le n \tag{11}$$

is equal to M given by (10); those integers are possible exponent accompanying k independent variables of the polynomial of order n. Notice that two solutions that differ by the order must be regarded as different. Since $M = (n+k)!\{k!n!\}^{-1}$ the strong upper-bound for the order of the polynomial will be $n + 1 \le \{k!N\}^{1/k}$.

5 Evolutionary algorithm for solution of Problem B

Having so many unknowns we can suggest a hierarchial optimization procedure to determine the form of the function to be used to approximate wanted functional Ψ with given accuracy ϵ on the set TRE. It can be as follows.

1. find two natural numbers k and n, both smaller than N, and
2. find M real-valued coefficients[5] $c_0, c_1, \ldots, c_{M-1}$, where M is given by (10), of a polynomial $W(z_1, z_2, \ldots, z_k)$ of k variables of order n, and
3. find $M-1$ aggregates of nonnegative integer exponents $\{m_{1j}, m_{2j}, \ldots, m_{kj}\}$, satisfying the inequality $m_{1j} + m_{2j} + \cdots + m_{kj} \le n$, with $j = 1, 2, \ldots, M-1$, each of them giving the power of the corresponding k variables $z_1, z_2, \ldots, z_k$, and hence appearing in the following representation of the polynomial W of order n

$$W(z_1, z_2, \ldots, z_k) = c_0 + \sum_{j=1}^{M-1} c_j z_1^{m_{1j}} z_2^{m_{2j}} \ldots . z_k^{m_{kj}} \ , \tag{12}$$

[5] Some of them, of course, can be equal to zero.

4. find k continuous and linear functionals $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathcal{D}$ on $\mathcal{R}$, and
5. everything should be done under the condition (compare (9))

$$\max_{1 \le p \le N} |W \circ \Phi(A_p) - r_p| \le \epsilon \ , \ \text{where } (A_p, r_p) \in \mathsf{TRE} \ , \tag{13}$$

   where $W \circ \Phi$ denotes the composition of the polynomial W with k functionals $\varphi_1, \varphi_2, \ldots, \varphi_k$, i.e. the functional φ_i substitutes the variable z_i in the representation (12), with $i = 1, 2, \ldots, k$.

To solve Problem B we have designed the evolutionary algorithm presented below. If we look once more at items 1.–5. above then we can see that finding under the constraint 5. three types of objects, namely those appearing in 2. 3. and 4. can be sufficient, if we additionally require some extra constraints between the numbers M, k and n. Taking this into consideration the genotype encoding searching solution (shown in Fig. 2) is composed of three types of chromosome:

1. g-chromosome consisting of N real numbers,
2. m-chromosome composed of $N-1$ nonnegative integers,
3. ϕ-chromosome being a pair of real value functions of bounded variation defined on $I = [0, 1]$.

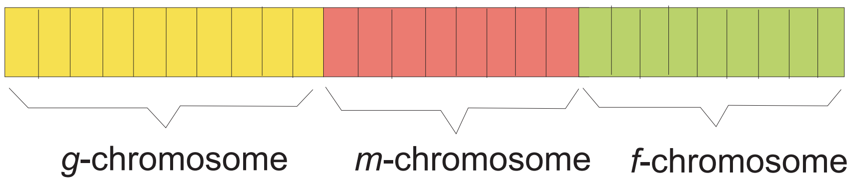

Fig. 2. The schema of the genotype

More precisely, g-chromosome represents coefficients $c_0, c_1, \ldots, c_{M-1}$, where M is given by (10) of polynomial $W(z_1, z_2, \ldots, z_k)$ of k variables of order n. This chromosome is schematically shown in Fig. 3. The first gene refers to the value M which is less or equal to N and causes that during decoding of the chromosome only M genes are active to create the polynomial. The next chromosome is presented in

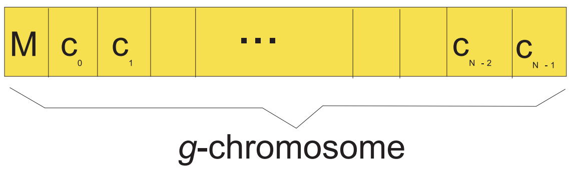

Fig. 3. The schema of the g-chromosome

Fig. 4. It is composed of genes encoding nonnegative integer exponents $\{m_{1j}, m_{2j}, \ldots, m_{kj}\}$, satisfying the inequality $m_{1j} + m_{2j} + \cdots + m_{kj} \le n$, with $j = 1, 2, \ldots, M-1$, each of them giving the power of the corresponding k variables $z_1, z_2, \ldots, z_k$. As in the g-chromosome, the length of m-chromosome is equal to

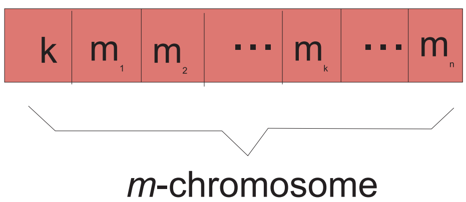

Fig. 4. The schema of the m-chromosome

N, but after its decoding the value of the first gene informs that only k exponents are active. It assigns also the number of active genes in the ϕ-chromosome, which is shown in the Fig. 5. Here ϕ-chromosome

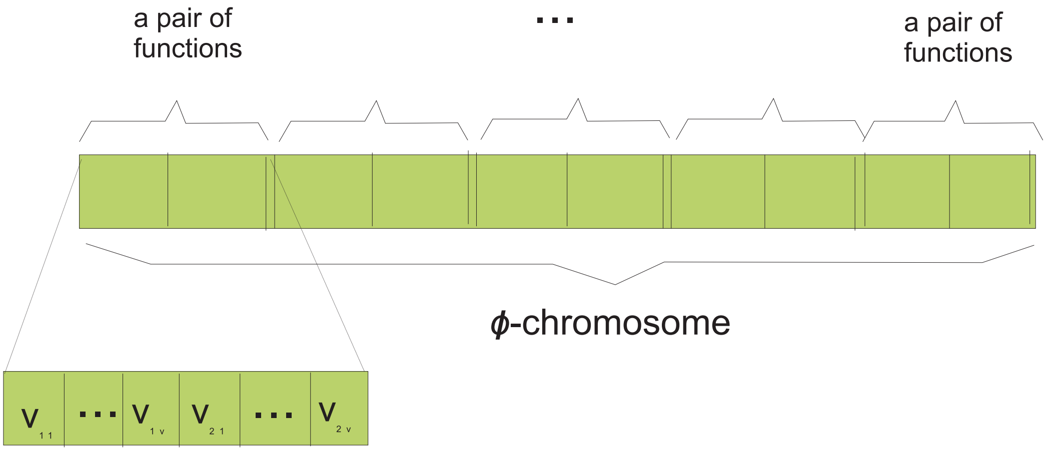

Fig. 5. The schema of the ϕ-chromosome

after decoding gives k continuous and linear functionals. The functional φ_i substitutes the variable z_i in the representation (12).

This chromosome has a more complex structure comparing with the presented above. It is the consequence of the assumed representation of linear and bounded functionals (6). Due to the general properties of a function of bounded variation (cf. [11, 21]) according to which it may possess a countable number of jumps, we have assumed that there is a maximal number of such jumps situated at points of function discontinuity in the interval $I = [0, 1]$, and is equal to v. Hence for the pair of such functions (h_1, h_2) those points are enumerated by s_{q1}, s_{q2}, $\ldots$,$s_{qv}, q = 1, 2$. They may be differ for different functions.

At each discontinuity point $s_{qk}, k \leq v$ one should give two values: the left–handed limit value $h_q(s_{ik}^-)$ and the right-handed limit value $h_q(s_{ik}^+)$ of the corresponding function h_q. Assuming, for simplicity, that between each pair of the consequent discontinuity points, i.e. between s_{qk} and $s_{q(k+1)}$, each function h_q is affine, i.e. of the form $a_k s + b_k$, with $k \leq v - 1$, and two reals a_k and b_k, where s denotes the independent variable varying in $[0, 1]$, one can uniquely determine from those limit values that pair of constants a_k and b_k. It is obvious that in the general case those constant are different for different k. Consequently, in a typical ϕ-chromosome a pair of functions are represented by the collection $\{v_{11}, v_{12}, \ldots, v_{1v}, v_{21}, v_{22}, \ldots, v_{2v}\}$, where the typical entry $v_{qk}, q = 1, 2, 1 \leq k \leq v$, forms three real numbers $(s_{qk}, h_q(s_{ik}^-), h_q(s_{ik}^+))$, in Fig. 5.

Two main operations performed on the genotypes are: mutation and crossover. Uniform crossover is chosen where information is exchanged between corresponding chromosomes. It means the cutting points can lie between chromosomes, only. Mutation is separately defined depending on the type of gene it refers to. For the genes being natural number it is realized as a substitution of the actual value by a new one; this value is assigned randomly. For genes containing real number x its new value is defined by randomly chosen small real value Δ_x that can be positive or negative.

$$x_{new} = x_{act} + \Delta_x \tag{14}$$

To finish the description of the genetic operations it is necessary to introduce the repairing operation. In the case when the new values of genes (received after mutation) are besides of the domain they must be repaired. New values are defined by random choice, but once again it must be verified whether the the appropriate conditions are satisfied. In order to apply evolutionary algorithm the individuals in each generation have to be evaluated. In the proposed approach they are evaluated on the phenotype level, that needs to decode the genotype. Then the value of fitness function is calculated; it forms a basis for the last genetic operation, namely, selection. It can be, for example, the proportional selection.

The fitness function is the natural consequence of the condition (9) that should be satisfied to obtain a solution. It is formulated as:

$$eval(W_{g,m}, \Phi) = \{1 + \sum_{p=1}^{N} |W_{g,m} \circ \Phi(A_p) - r_p|\}^{-1} \ . \tag{15}$$

After defining all necessary components of the proposed evolutionary algorithm, its performance can be described in steps typical for classical genetic algorithm. At the beginning the necessary parameters as crossover probability -p_{cross} and mutation probability p_{mu} as well the number *card* of individuals in the population should be assigned by the user. Then the algorithm proceeds as follows:

1. Create randomly *card* individuals in the population.
2. Evaluate them using fitness function given by (15).
3. While the stopping criteria is not satisfied (test the condition given by (9)).
   Begin
      select individuals to the new generation;
      perform genetic operations;
      calculate fitness function (15);
   End
4. Stop

Obtained solution to Problem B should be checked with different set of data, called the test set TEST. The error on that new set should not exceed too much the primitive value of ϵ, say $d\epsilon$, the factor d less than 2 is acceptable. In further paper results of implementation of the presented algorithm will be reported.

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