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An Approximation Branch-and-Bound Algorithm for Fuzzy Bilevel Decision Making Problems

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Abstract. Organizational decision making often involves two decision levels. When the leader at the upper level attempts to optimize his/her objective, the follower at the lower level tries to find an optimized strategy according to each of possible decisions made by the leader. Furthermore, such bilevel decision making may involve uncertain parameters which appear either in the objective functions or constraints of the leader or the follower. Following our previous work on fuzzy bilevel decision making, this study proposes a solution concept and related theorems for general- fuzzy-number based fuzzy parameter bilevel programming problems. It then develops an approximation Branch-and-bound algorithm to solve the proposed problem.

Key words: Bilevel programming, Branch-and-bound algorithm, Fuzzy sets, Optimization, Decision making

1 Introduction

Bilevel decision making (also called bilevel programming, BLP) techniques, first introduced by Von Stackelberg [18], have been developed for mainly solving decentralized planning problems with decision makers in a hierarchical organization [3, 4, 7, 8, 19]. Decision maker at the upper level is termed as the leader, and at the lower level, the follower. Each decision maker (leader or follower) tries to optimize his/her own objective function, but the decision of each level affects the objective value of the other level [12].

Bilevel decision making theory and technology have been applied with remarkable success in different domains, for example, decentralized resource planning, electric power market, logistics, civil engineering, chemical engineering and road network management [1–3, 11, 12]. The majority of research on BLP has centered on the linear version of the problem, i.e., linear BLP problems. A set of approaches and algorithms have been well developed such as well known Kuhn-Tucker approach [4], Kth-best approach [6] and Branch-and-bound algorithm [5, 9]. However, existing BLP approaches mainly suppose the situation in which the objective functions and constraints are characterized with precise parameters. Therefore, the parameters are required to be fixed at the some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters in the problem-formulation process. It has been observed that, in most real-world situations, for example, power market, the possible values of these parameters are often only imprecisely or ambiguously known to the experts who establish this model. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets [20]. A BLP problem in which the parameters, either in objective functions or in constrains of the leader or the follower, are described by fuzzy values is called a fuzzy bilevel programming (FBLP) or a fuzzy bilevel decision making (FBLDM) problem in the study.

Based on the definitions given by Bard et al. [4, 5] and Lai et al [10, 17] BLP problems can be described into two situations: cooperative and noncooperative. The cooperative BLP proposed by Lai assumes the leader and the follower making their decisions under a cooperative relationship, while a noncooperative BLP problem proposed by Bard assumes that the follower reacts the leader's decision in a totally personally optimal way. Both situations can be happened in real-world decision-making practice. The FBLP problem was first researched by Sakawa et al. in 2000 [13]. Sakawa et al. formulated cooperative FBLP problem and proposed a fuzzy programming approach for solving the problem. In the approach, Sakawa introduced the concepts of α-bilevel programming based on the basis of fuzzy number α-level sets. Our research deals with noncooperative FBLP problems. Based on the extended solution concept and related theorems of BLP [14–16], we first solve FBLP problem with a specialized forms of membership functions, triangular form, in the fuzzy parameters [21, 22]. However, this may restrict the use of other forms of membership functions to describe the parameters in modeling an FBLP problem. This paper extends our previous research by allowing any form of membership functions to describe the fuzzy parameters in a FBLP model. It develops an approximation Branch-and-bound algorithm to solve the general FBLP problem in an uncooperative environment.

Following the introduction, Section Section 2 reviews related the model and theorems for fuzzy linear bilevel programming. A general-fuzzy-number based approximation Branch-and-bound algorithm for solving FBLP problems are presented in Section 3. Conclusions and further study are discussed in Section 4.

2 A Fuzzy Linear Bilevel Programming Model

This section gives related definitions and theories for general-fuzzy-number based FBLP model.

Consider the following fuzzy linear bilevel programming problem: For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to F^*(R)$, and $f : X \times Y \to F^*(R)$,

$$\min_{x\in X} F(x,y) = \tilde{c}_1 x + \tilde{d}_1 y \tag{1a}$$

$$\text{subject to } \tilde{A}_1 x + \tilde{B}_1 y \leqq \tilde{b}_1 \tag{1b}$$

$$\min_{y\in Y} f(x,y) = \tilde{c}_2 x + \tilde{d}_2 y \tag{1c}$$

$$\text{subject to } \tilde{A}_2 x + \tilde{B}_2 y \leqq \tilde{b}_2 \tag{1d}$$

where $\tilde{c}_1, \tilde{c}_2 \in F^*(R^n)$, $\tilde{d}_1, \tilde{d}_2 \in F^*(R^m)$, $\tilde{b}_1 \in F^*(R^p)$, $\tilde{b}_2 \in F^*(R^q)$, $\tilde{A}_1 = \left(\tilde{a}_{ij}\right)_{p\times n}$, $\tilde{a}_{ij} \in F^*(R)$, $\tilde{B}_1 = \left(\tilde{b}_{ij}\right)_{p\times m}$, $\tilde{b}_{ij} \in F^*(R)$, $\tilde{A}_2 = \left(\tilde{e}_{ij}\right)_{q\times n}$, $\tilde{e}_{ij} \in F^*(R)$, $\tilde{B}_2 = \left(\tilde{s}_{ij}\right)_{q\times m}$, $\tilde{s}_{ij} \in F^*(R)$.

Associated with the (FLBLP) problem, we now consider the following (MLBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to F^*(R)$, and $f : X \times Y \to F^*(R)$,

$$\begin{aligned} &\min_{x\in X} (F(x,y))^L_\lambda = c_{1\lambda}^L x + d_{1\lambda}^L y, \quad \lambda \in [0,1] \\ &\min_{x\in X} (F(x,y))^R_\lambda = c_{1\lambda}^R x + d_{1\lambda}^R y, \quad \lambda \in [0,1] \end{aligned} \tag{2a}$$

$$\text{subject to } A_{1\lambda}^L x + B_{1\lambda}^L y \leqq b_{1\lambda}^L, A_{1\lambda}^R x + B_{1\lambda}^R y \leqq b_{1\lambda}^R, \quad \lambda \in [0,1] \tag{2b}$$

$$\begin{aligned} &\min_{y\in Y} (f(x,y))^L_\lambda = c_{2\lambda}^L x + d_{2\lambda}^L y, \quad \lambda \in [0,1] \\ &\min_{y\in Y} (f(x,y))^R_\lambda = c_{2\lambda}^R x + d_{2\lambda}^R y, \quad \lambda \in [0,1] \end{aligned} \tag{2c}$$

$$\text{subject to } A_{2\lambda}^L x + B_{2\lambda}^L y \leqq b_{2\lambda}^L, A_{2\lambda}^R x + B_{2\lambda}^R y \leqq b_{2\lambda}^R, \quad \lambda \in [0,1] \tag{2d}$$

where $c_{1\lambda}^L, c_{1\lambda}^R, c_{2\lambda}^L, c_{2\lambda}^R \in R^n$, $d_{1\lambda}^L, d_{1\lambda}^R, d_{2\lambda}^L, d_{2\lambda}^R \in R^m$, $b_{1\lambda}^L, b_{1\lambda}^R \in R^p$, $b_{2\lambda}^L, b_{2\lambda}^R \in R^q$, $A_{1\lambda}^L = \left(a_{ij\lambda}^L\right)$, $A_{1\lambda}^R = \left(a_{ij\lambda}^R\right) \in R^{p\times n}$, $B_{1\lambda}^L = \left(b_{ij\lambda}^L\right)$, $B_{1\lambda}^R = \left(b_{ij\lambda}^R\right) \in R^{p\times m}$, $A_{2\lambda}^L = \left(e_{ij\lambda}^L\right)$, $A_{2\lambda}^R = \left(e_{ij\lambda}^R\right) \in$, ${}^{q\times n}$, $B_{2\lambda}^L = \left(s_{ij\lambda}^L\right)$, $B_{2\lambda}^R = \left(s_{ij\lambda}^R\right) \in R^{q\times m}$.

Theorem 1. *[23] Let (x^*, y^*) be the solution of the (MLBLP) problem (2). Then it is also a solution of the (FLBLP) problem defined by (1).*

Theorem 2. *[23] For $x \in X \subset R^n$, $y \in Y \subset R^m$, If all the fuzzy coefficients $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_i$ and $\tilde{d}_i$ have trapezoidal membership functions of the (MLBLP) problem (1).*

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z^L_\beta \\ \frac{\alpha-\beta}{z^L_\alpha - z^L_\beta}\left(t - z^L_\beta\right) + \beta & z^L_\beta \leqq t < z^L_\alpha \\ \alpha & z^L_\alpha \leqq t < z^R_\alpha \\ \frac{\alpha-\beta}{z^R_\beta - z^R_\alpha}\left(-t + z^R_\beta\right) + \beta & z^R_\alpha \leqq t \leqq z^R_\beta \\ 0 & z^R_\beta \; y < t \end{cases}, \tag{3}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$ respectively. Then, it is the solution of the problem (1) that $(x^*, y^*) \in R^n \times R^m$ satisfying

$$\begin{aligned} &\min_{x\in X} (F(x,y))^L_\alpha = c_{1\alpha}^L x + d_{1\alpha}^L y, \\ &\min_{x\in X} (F(x,y))^R_\alpha = c_{1\alpha}^R x + d_{1\alpha}^R y, \\ &\min_{x\in X} (F(x,y))^L_\beta = c_{1\beta}^L x + d_{1\beta}^L y, \\ &\min_{x\in X} (F(x,y))^R_\beta = c_{1\beta}^R x + d_{1\beta}^R y, \end{aligned} \tag{4a}$$

$$\begin{aligned} \textit{subject to } & A_{1\alpha}^L x + B_{1\alpha}^L y \leqq b_{1\alpha}^L, \\ & A_{1\alpha}^R x + B_{1\alpha}^R y \leqq b_{1\alpha}^R, \\ & A_{1\beta}^L x + B_{1\beta}^L y \leqq b_{1\beta}^L, \\ & A_{1\beta}^R x + B_{1\beta}^R y \leqq b_{1\beta}^R, \end{aligned} \tag{4b}$$

$$\begin{aligned}
&\min_{y\in Y}(f(x,y))^L_\alpha = c_2{}^L_\alpha x + d_2{}^L_\alpha y,\\
&\min_{y\in Y}(f(x,y))^L_\alpha = c_2{}^L_\alpha x + d_2{}^L_\alpha y,\\
&\min_{y\in Y}(f(x,y))^L_\beta = c_2{}^L_\beta x + d_2{}^L_\beta y,\\
&\min_{y\in Y}(f(x,y))^R_\beta = c_2{}^R_\beta x + d_2{}^R_\beta y,
\end{aligned} \tag{4c}$$

$$\begin{aligned}
\textit{subject to } & A_2{}^L_\alpha x + B_2{}^L_\alpha y \leqq b_2{}^L_\alpha,\\
& A_2{}^L_\alpha x + B_2{}^L_\alpha y \leqq b_2{}^L_\alpha,\\
& A_2{}^L_\beta x + B_2{}^L_\beta y \leqq b_2{}^L_\beta,\\
& A_2{}^R_\beta x + B_2{}^R_\beta y \leqq b_2{}^R_\beta.
\end{aligned} \tag{4d}$$

Theorem 3. *[23] For $x \in X \subset R^n$, $y \in Y \subset R^m$, If all the fuzzy coefficients $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$ have trapezoidal membership functions of the (MLBLP) problem (1).*

$$\mu_{\tilde{z}}(t) = \begin{cases}
0 & t < z^L_{\alpha_0}\\
\frac{\alpha_1-\alpha_0}{z^L_{\alpha_1}-z^L_{\alpha_0}}\left(t - z^L_{\alpha_0}\right) + \alpha_0 & z^L_{\alpha_0} \leqq t < z^L_{\alpha_1}\\
\frac{\alpha_1-\alpha_0}{z^L_{\alpha_2}-z^L_{\alpha_1}}\left(t - z^L_{\alpha_1}\right) + \alpha_1 & z^L_{\alpha_1} \leqq t < z^L_{\alpha_2}\\
\cdots & \cdots\\
\alpha & z^L_{\alpha_n} \leqq t < z^R_{\alpha_n}\\
\frac{\alpha_n-\alpha_{n-1}}{z^R_{\alpha_{n-1}}-z^R_{\alpha_n}}\left(-t + z^R_{\alpha_{n-1}}\right) + \alpha_{n-1} & z^R_{\alpha_n} \leqq t < z^R_{\alpha_{n-1}}\\
\cdots & \cdots\\
\frac{\alpha_0-\alpha_1}{z^R_{\alpha_1}-z^R_{\alpha_0}}\left(-t + z^R_{\alpha_0}\right) + \alpha_0 & z^R_{\alpha_1} \leqq t \leqq z^R_{\alpha_0}\\
0 & z^R_{\alpha_0} < t
\end{cases}, \tag{5}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$respectively. Then, it is the solution of the problem (1) that $(x^*, y^*) \in R^n \times R^m$ satisfying

$$\begin{aligned}
&\min_{x\in X}(F(x,y))^L_{\alpha_0} = c_1{}^L_{\alpha_0} x + d_1{}^L_{\alpha_0} y,\\
&\qquad\vdots\\
&\min_{x\in X}(F(x,y))^L_{\alpha_n} = c_1{}^L_{\alpha_n} x + d_1{}^L_{\alpha_n} y,\\
&\min_{x\in X}(F(x,y))^R_{\alpha_0} = c_1{}^R_{\alpha_0} x + d_1{}^R_{\alpha_0} y,\\
&\qquad\vdots\\
&\min_{x\in X}(F(x,y))^R_{\alpha_n} = c_1{}^R_{\alpha_n} x + d_1{}^R_{\alpha_n} y,
\end{aligned} \tag{6a}$$

$$\begin{aligned}
\textit{subject to } & A_1{}^L_{\alpha_0} x + B_1{}^L_{\alpha_0} y \leqq b_1{}^L_{\alpha_0},\\
&\qquad\vdots\\
& A_1{}^L_{\alpha_n} x + B_1{}^L_{\alpha_n} y \leqq b_1{}^L_{\alpha_n},\\
& A_1{}^R_{\alpha_0} x + B_1{}^R_{\alpha_0} y \leqq b_1{}^R_{\alpha_0},\\
&\qquad\vdots\\
& A_1{}^R_{\alpha_n} x + B_1{}^R_{\alpha_n} y \leqq b_1{}^R_{\alpha_n},
\end{aligned} \tag{6b}$$

$$\begin{aligned}
&\min_{y\in Y}(f(x,y))^L_{\alpha_0} = c_2{}^L_{\alpha_0}x + d_2{}^L_{\alpha_0}y,\\
&\quad\vdots\\
&\min_{y\in Y}(f(x,y))^L_{\alpha_n} = c_2{}^L_{\alpha_n}x + d_2{}^L_{\alpha_n}y,\\
&\min_{y\in Y}(f(x,y))^R_{\alpha_0} = c_2{}^R_{\alpha_0}x + d_2{}^R_{\alpha_0}y,\\
&\quad\vdots\\
&\min_{y\in Y}(f(x,y))^R_{\alpha_n} = c_2{}^R_{\alpha_n}x + d_2{}^R_{\alpha_n}y,
\end{aligned} \tag{6c}$$

$$\begin{aligned}
\textit{subject to } &A_2{}^L_{\alpha_0}x + B_2{}^L_{\alpha_0}y \leqq b_2{}^L_{\alpha_0},\\
&\quad\vdots\\
&A_2{}^L_{\alpha_n}x + B_2{}^L_{\alpha_n}y \leqq b_2{}^L_{\alpha_n},\\
&A_2{}^R_{\alpha_0}x + B_2{}^R_{\alpha_0}y \leqq b_2{}^R_{\alpha_0},\\
&\quad\vdots\\
&A_2{}^R_{\alpha_n}x + B_2{}^R_{\alpha_n}y \leqq b_2{}^R_{\alpha_n}.
\end{aligned} \tag{6d}$$

We note

$$\overline{A}_1x + \overline{B}_1y \leqq \overline{b}_1 \tag{6b$'$}$$

$$\overline{A}_2x + \overline{B}_2y \leqq \overline{b}_2 \tag{6d$'$}$$

where

$$\overline{A}_1 = \begin{pmatrix} A_1{}^L_{\alpha_0}\\ \vdots\\ A_1{}^L_{\alpha_n}\\ A_1{}^R_{\alpha_0}\\ \vdots\\ A_1{}^R_{\alpha_n} \end{pmatrix}, \overline{A}_2 = \begin{pmatrix} A_2{}^L_{\alpha_0}\\ \vdots\\ A_2{}^L_{\alpha_n}\\ A_2{}^R_{\alpha_0}\\ \vdots\\ A_2{}^R_{\alpha_n} \end{pmatrix}, \overline{B}_1 = \begin{pmatrix} B_1{}^L_{\alpha_0}\\ \vdots\\ B_1{}^L_{\alpha_n}\\ B_1{}^R_{\alpha_0}\\ \vdots\\ B_1{}^R_{\alpha_n} \end{pmatrix}, \overline{B}_2 = \begin{pmatrix} B_2{}^L_{\alpha_0}\\ \vdots\\ B_2{}^L_{\alpha_n}\\ B_2{}^R_{\alpha_0}\\ \vdots\\ B_2{}^R_{\alpha_n} \end{pmatrix},$$

$$\overline{b}_1 = \begin{pmatrix} b_1{}^L_{\alpha_0}\\ \vdots\\ b_1{}^L_{\alpha_n}\\ b_1{}^R_{\alpha_0}\\ \vdots\\ b_1{}^R_{\alpha_n} \end{pmatrix}, \overline{b}_2 = \begin{pmatrix} b_2{}^L_{\alpha_0}\\ \vdots\\ b_2{}^L_{\alpha_n}\\ b_2{}^R_{\alpha_0}\\ \vdots\\ b_2{}^R_{\alpha_n} \end{pmatrix}.$$

Then we can re-write (6) by using

$$\begin{aligned} &\min_{x\in X}(F(x,y))^L_{\alpha_0} = c_1{}^L_{\alpha_0}x + d_1{}^L_{\alpha_0}y,\\ &\quad\vdots\\ &\min_{x\in X}(F(x,y))^L_{\alpha_n} = c_1{}^L_{\alpha_n}x + d_1{}^L_{\alpha_n}y,\\ &\min_{x\in X}(F(x,y))^R_{\alpha_0} = c_1{}^R_{\alpha_0}x + d_1{}^R_{\alpha_0}y,\\ &\quad\vdots\\ &\min_{x\in X}(F(x,y))^R_{\alpha_n} = c_1{}^R_{\alpha_n}x + d_1{}^R_{\alpha_n}y, \end{aligned} \tag{6a$'$}$$

$$\text{subject to } \bar{A}_1x + \bar{B}_1y \leqq \bar{b}_1, \tag{6b$'$}$$

$$\begin{aligned} &\min_{y\in Y}(f(x,y))^L_{\alpha_0} = c_2{}^L_{\alpha_0}x + d_2{}^L_{\alpha_0}y,\\ &\quad\vdots\\ &\min_{y\in Y}(f(x,y))^L_{\alpha_n} = c_2{}^L_{\alpha_n}x + d_2{}^L_{\alpha_n}y,\\ &\min_{y\in Y}(f(x,y))^R_{\alpha_0} = c_2{}^R_{\alpha_0}x + d_2{}^R_{\alpha_0}y,\\ &\quad\vdots\\ &\min_{y\in Y}(f(x,y))^R_{\alpha_n} = c_2{}^R_{\alpha_n}x + d_2{}^R_{\alpha_n}y, \end{aligned} \tag{6c$'$}$$

$$\text{subject to } \bar{A}_2x + \bar{B}_2y \leqq \bar{b}_2. \tag{6d$'$}$$

To solve the problem (7), we can use the method of weighting to this problem, such that it is the following problem:

$$\min_{x\in X}(F(x,y)) = \sum_{i=0}^{n}\left(\left(c_1{}^L_{\alpha_i}x + d_1{}^L_{\alpha_i}y\right) + \left(c_1{}^R_{\alpha_i}x + d_1{}^R_{\alpha_i}y\right)\right) \tag{7a}$$

$$\text{subject to } \bar{A}_1x + \bar{B}_1y \leqq \bar{b}_1, \tag{7b}$$

$$\min_{y\in Y}(f(x,y)) = \sum_{i=0}^{n}\left(\left(c_2{}^L_{\alpha_i}x + d_2{}^L_{\alpha_i}y\right) + \left(c_2{}^R_{\alpha_i}x + d_2{}^R_{\alpha_i}y\right)\right) \tag{7c}$$

$$\text{subject to } \bar{A}_2x + \bar{B}_2y \leqq \bar{b}_2. \tag{7d}$$

Theorem 4. *[23] For $x \in X \subset R^n$, $y \in Y \subset R^m$, If all the fuzzy coefficients $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$ have trapezoidal membership functions of the (MLBLP) problem (1).*

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z^L_{\alpha_0}\\ \frac{\alpha_1-\alpha_0}{z^L_{\alpha_1}-z^L_{\alpha_0}}\left(t - z^L_{\alpha_0}\right) + \alpha_0 & z^L_{\alpha_0} \leqq t < z^L_{\alpha_1}\\ \frac{\alpha_1-\alpha_0}{z^L_{\alpha_2}-z^L_{\alpha_1}}\left(t - z^L_{\alpha_1}\right) + \alpha_1 & z^L_{\alpha_1} \leqq t < z^L_{\alpha_2}\\ \cdots & \cdots\\ \alpha & z^L_{\alpha_n} \leqq t < z^R_{\alpha_n}\\ \frac{\alpha_n-\alpha_{n-1}}{z^R_{\alpha_{n-1}}-z^R_{\alpha_n}}\left(-t + z^R_{\alpha_{n-1}}\right) + \alpha_{n-1} & z^R_{\alpha_n} \leqq t < z^R_{\alpha_{n-1}}\\ \cdots & \cdots\\ \frac{\alpha_0-\alpha_1}{z^R_{\alpha_1}-z^R_{\alpha_0}}\left(-t + z^R_{\alpha_0}\right) + \alpha_0 & z^R_{\alpha_1} \leqq t \leqq z^R_{\alpha_0}\\ 0 & z^R_{\alpha_0} < t \end{cases}, \tag{8}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_i$ and $\tilde{d}_i$respectively. Then a necessary and sufficient condition that (x^*, y^*) solves the fuzzy linear BLP problem (1) is that there exist (row) vectors u^*, $v^*$and w^* such that $(x^*, y^*, u^*, v^*, w^*)$

solves:

$$\min_{x\in X}(F(x,y)) = \sum_{i=0}^{n}\left(c_{1\alpha_i}^{L}x + d_{1\alpha_i}^{L}y\right) + \sum_{i=0}^{n}\left(c_{1\alpha_i}^{R}x + d_{1\alpha_i}^{R}y\right) \tag{9a}$$

$$\text{subject to } \bar{A}_1 x + \bar{B}_1 y \leqq \bar{b}_1, \tag{9b}$$

$$\bar{A}_2 x + \bar{B}_2 y \leqq \bar{b}_2, \tag{9c}$$

$$\begin{aligned} & u\left(\sum_{i=0}^{n} B_{1\alpha_i}^{L} + \sum_{i=0}^{n} B_{1\alpha_i}^{R}\right) + v\left(\sum_{i=0}^{n} B_{2\alpha_i}^{L} + \sum_{i=0}^{n} B_{2\alpha_i}^{R}\right) - w \\ & \qquad = -\left(\sum_{i=0}^{n} d_{2\alpha_i}^{L} + \sum_{i=0}^{n} d_{2\alpha_i}^{R}\right) \end{aligned} \tag{9d}$$

$$\begin{aligned} & u\left(\left(\sum_{i=0}^{n} b_{1\alpha_i}^{L} + \sum_{i=0}^{n} b_{1\alpha_i}^{R}\right) - \left(\sum_{i=0}^{n} A_{1\alpha_i}^{L} + \sum_{i=0}^{n} A_{1\alpha_i}^{R}\right)x \right.\\ & \qquad \left. - \left(\sum_{i=0}^{n} B_{1\alpha_i}^{L} + \sum_{i=0}^{n} B_{1\alpha_i}^{R}\right)y\right) + v\left(\left(\sum_{i=0}^{n} b_{2\alpha_i}^{L} + \sum_{i=0}^{n} b_{2\alpha_i}^{R}\right)\right. \\ & \qquad \left. - \left(\sum_{i=0}^{n} A_{2\alpha_i}^{L} + \sum_{i=0}^{n} A_{2\alpha_i}^{R}\right)x - \left(\sum_{i=0}^{n} B_{2\alpha_i}^{L} + \sum_{i=0}^{n} B_{2\alpha_i}^{R}\right)y\right) \\ & \qquad + wy = 0 \end{aligned} \tag{9e}$$

$$x \geqslant 0, y \geqslant 0, u \geqslant 0, v \geqslant 0, w \geqslant 0. \tag{9f}$$

Based on Theorem 3, we present an approximation Branch-and-bound approach for solving the fuzzy linear bilevel programming shown in (1).

3 An Approximation Branch-and-Bound Algorithm

We first write all the inequalities (except of the leader's variables) of (7a)-(7d) as $g_i(x,y) \geqslant 0, i = 1,\dots,p+q+m$, and note that complementary slackness simply means $u_i g_i(x,y) = 0 (i = 1,\dots,p+q+m)$. Now we suppress the complementary term and solve the resulted linear sub-problem. At each time of iteration the condition (9e) is checked. If it is satisfied, the corresponding point is in the inducible region and hence a potential solution to (7). Otherwise, a Branch-and-bound scheme is used to implicitly examine all combinations of the complementarities slackness. A flowchart of the basic idea is shown in Figure 3.1 to describe the approximation Branch-and-bound algorithm.

Based on this flowchart, we give some notations for describing the details of the approximation Branch-and-bound algorithm.

Let $W = \{1,\dots,p+q+m\}$ be the index set for the terms in (9), $\bar{F}$ be the incumbent upper bound on the objective function of the leader. At the k-th level of an search tree we define a subset of indices$W_k \subset W$, and a path P_k corresponding to an assignment of either $u_i = 0$ or $g_i = 0$ for $i \in W_k$. Now let

$$\begin{aligned} S_k^+ &= \{i : i \in W_k, u_i = 0\} \\ S_k^- &= \{i : i \in W_k, g_i = 0\} \\ S_k^0 &= \{i : i \notin W_k\}. \end{aligned}$$

For $i \in S_k^0$, the variables u_i or g_i are free to assume any nonnegative value in the solution of (9) with (9e) omitted, so complementary slackness will not necessarily be satisfied.

By using these notations we give all steps of the approximation Branch-to-bound algorithm.

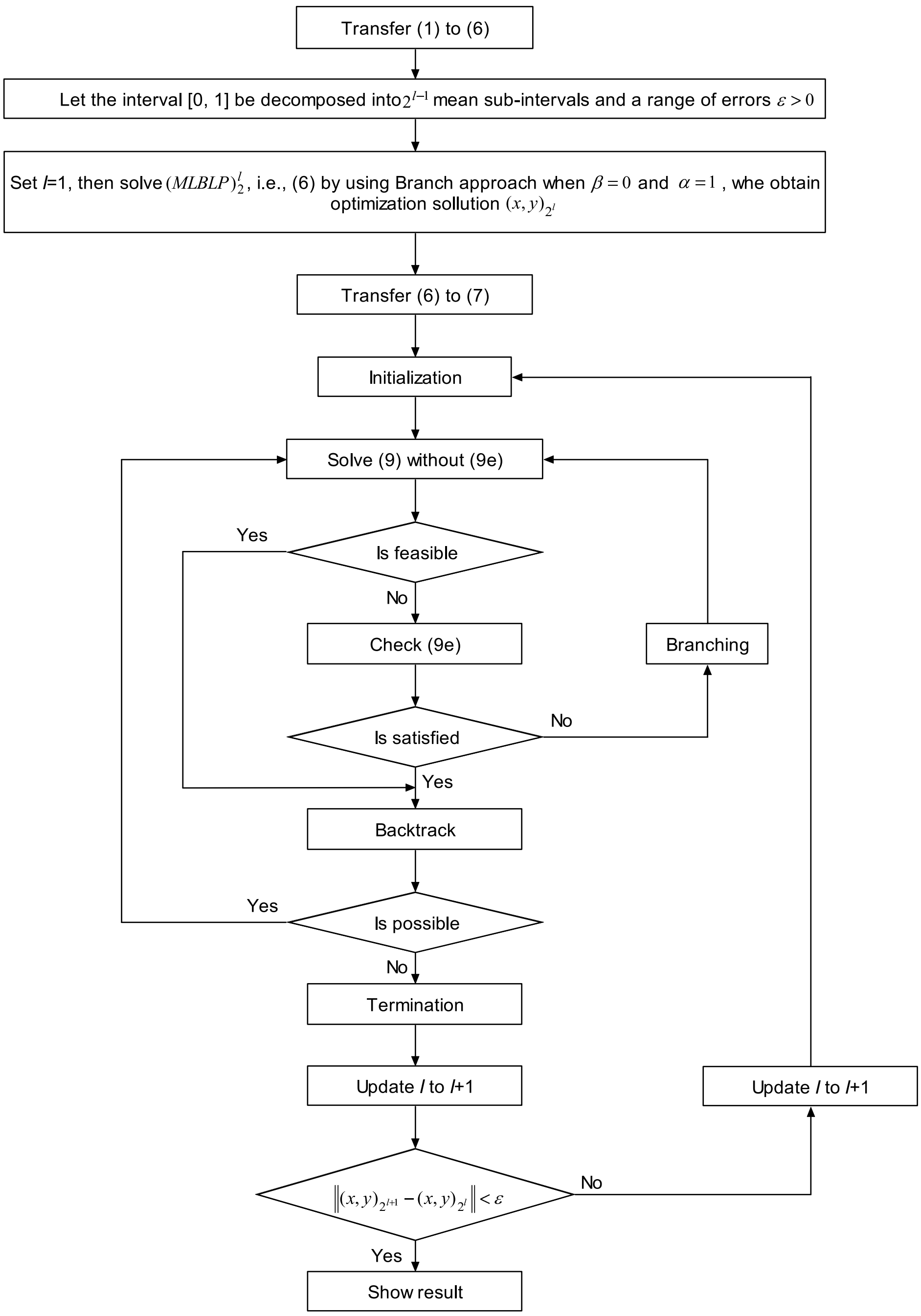

Fig. 1. A flowchart for the main ideas of the extend branch-and-bound algorithm

An Approximation Branch-and-bound algorithm for FBLP problems

Step 1	The problem (2.1) is transferred to the problem (6)
Step 2	Let the interval [0, 1] be decomposed into 2^{l-1} mean sub-intervals with $(2^{l-1}+1)$ nodes $\lambda_i\left(i = 0, \cdots, 2^{l-1}\right)$ which are arranged in the order of0 $= \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$ and a range of errors $\varepsilon > 0$.
Step 3	Set $l = 1$, then solve $(\text{MLBLP})_2^l$, i.e. (6) by using Branch approach when $\beta = 0$ and $\alpha = 1$, we obtain optimization solution$(x, y)_{2^l}$.
Step 4	The problem (6) is transferred to the following linear BLP problem (7) by using method of weighting
Step 5	To solve the problem (7)
Step 5.0	(Initialization) Set $k = 0$, $S_k^+ = \phi$, $S_k^- = \phi$, $S_k^0 = \{1, \ldots, p + q + m\}$, and $\overline{F} = \infty$.
Step 5.1	(Iteration k) Set $u_i = 0$ for $i \in S_k^+$ and $g_i = 0$ for $i \in S_k^-$. It first attempts to solve (9) without (9e). If the resultant problem is infeasible, go to Step 5.5; otherwise, put $k \leftarrow k + 1$ and label the solution (x^k, y^k, u^k).
Step 5.2	(Fathoming) If $F(x^k, y^k) \geqslant \overline{F}$, then go to Step 5.5.
Step 5.3	(Branching) If$u_i^k g_i(x^k, y^k) = 0$, $i = 1, \ldots, p + q + m$, then go to Step 5.4. Otherwise select i for which $u_i^k g_i(x^k, y^k) \neq 0$ is the largest and label iti_1. Put$S_k^+ \leftarrow S_k^+ \cup \{i_1\}$, $S_k^0 \leftarrow S_k^0 \backslash \{i_1\}$, $S_k^- \leftarrow S_k^-$, append i_1 to P_k, and go to Step 5.1.
Step 5.4	(Updating) Let$\overline{F} \leftarrow F(x^k, y^k)$.
Step 5.5	(Backtracking) If no live node exists, go to Step 5.6. Otherwise branch to the newest live vertex and updateS_k^+,S_k^-,S_k^0 and P_k as discussed below. Go back to Step 5.1.
Step 5.6	(Termination) If$\overline{F} = \infty$, there is not feasible solution to $(\text{MLBLP})_2^l$. Otherwise, declare the feasible point associated with $\overline{F}$which is the optimal solution to $(\text{MLBLP})_2^l$.
Step 6	Solve $(\text{MLBLP})_2^{l+1}$ by Step 5.1 to Step 5.6 and we obtain optimization solution $(x, y)_{2^{l+1}}$.
Step 7	If$\left\|(x, y)_{2^{l+1}} - (x, y)_{2^l}\right\| < \varepsilon$, then the solution$(x^*, y^*)$of the fuzzy linear bilevel problem is $(x, y)_{2^{l+1}}$Otherwise, update l to $2l$ and go back to Step 6.
Step 8	Show the result of problem (1).

We give some explanations for these steps and their working process as follows.

After initialization, Step 5.1 is designed to find a new point which is potentially bilevel feasible. If no solution exists, or the solution does not offer an improvement over the incumbent (Step 5.2), the algorithm goes to Step 5.5 and backtracks.

Step 5.3 checks the value of $u_i^k g_i(x^k, y^k)$to determine if the complementary slackness conditions are satisfied. In practice, if $\left|u_i^k g_i\right| < 10^{-6}$ it is considered to be zero. Confirmation indicates that a feasible solution of a bilevel program has been found and at Step 5.4 the upper bound on the leader's objective function is updated. Alternatively, if the complementary slackness conditions are not satisfied, the term with the largest product is used at Step 5.3 to provide a branching variable. Branching is always completed on the Kuhn-Tucker multiplier [4].

At Step 5.5, the backtracking operation is performed. Note that a live node is one associated with a sub-problem that has not yet been fathomed at either Step 5.1 due to infeasibility or at Step 5.2 due to bounding, and whose solution violates at least one complementary slackness condition. To facilitate bookkeeping, the path P_k in the Branch-and-bound tree is represented by a vector, its dimension is the current depth of the tree. The order of the components of P_k is determined by their level in the tree. Indices only appear in P_k if they are in either S_k^+ or S_k^- with the entries underlined if they are in S_k^-. Because the algorithm always branches on a Kuhn-Tucker multiplier first, backtracking is accomplished by finding the rightmost non-underlined component if P_k, underlining it, and erasing all entries to the right. The erased entries are deleted from S_k^- and added to S_k^0.

4 Conclusions and Further Study

Uncertainty often occurs in bilevel decision making practice. Therefore, a BLP model often has fuzzy parameters in both its objective and constraint. An important issue involved in this situation is how to derive an optimal solution for the upper level's decision maker. This paper proposes a fuzzy number based the approximation

Branch-and-bound algorithm to solve such fuzzy bilevel decision problems. A cased based example is then given to illustrate the proposed approximation Branch-and-bound algorithm.

Further study on this topic includes the development of models and approaches for fuzzy bilevel multi-follower programming problems. In such a kind of problems, multiple followers are involved. The leader's decision will be affected not only by those followers' individual reactions but also by the relationships among these followers. As uncertain information could occur in the objectives and constraints of both the leader and his/her multiple followers, one of the challenges is how to get an optimal solution for the leader in the complex environment.

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